Mid-term Exam B.Math III Year (Differential Geometry) 2016

Attempt all questions. Each question carries 10 marks. Books and notes maybe consulted. Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises in the notes or Pressley's book, which haven't been solved in class must be proved in full if used.)

1. Consider the space curve:
$$\begin{aligned} c : (-1,1) &\to \mathbb{R}^3 \\ t &\mapsto \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right) \end{aligned}$$
Compute its Frenet frame $\{c'(t), n(t), b(t)\}$.

2. Let $c : [a,b] \to X$ be a smooth curve (parametrised by arc length) on an orientable surface $X \subset \mathbb{R}^3$. Let $\{c'(t), n(t), b(t)\}$ denote the Frenet frame of c, and let ν denote a unit normal field on X. For simplicity of notation, denote $\nu(c(t))$ by $\nu(t)$. Show that:
$$(n(t)\cdot\nu(t))\,(n(t)\cdot\nu'(t)) + (b(t)\cdot\nu(t))\,(b(t)\cdot\nu'(t)) = 0$$

3. Let $\lambda > 0$. Consider the helicoid surface, which is defined as:
$$X = \{(v\cos u,\ v\sin u,\ \lambda u) \in \mathbb{R}^3 :\ u \in \mathbb{R}, v \in \mathbb{R}\}$$
Find a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\{0\}$ is a regular value of f and $X = f^{-1}\{0\}$. Hence, or otherwise, find the tangent space to X at a point x as a subspace of $\mathbb{R}^3$.

4. Consider the ellipsoid:
$$X = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1\right\}$$
Let U be the open subset of $\mathbb{R}^3$ given by
$$U = \mathbb{R}^3 \setminus \{(x_1, 0, x_3) : x_1 \geq 0\}$$
Give a trigonometric chart (ϕ, V) for X with $\phi(V) = X \cap U$ analogous to the one for S^2 using spherical polar coordinates (latitudes and longitudes), and compute the first fundamental form of X in this chart using these coordinates.